# THEORETICAL DERIVATION OF SHAPE FUNCTIONS FOR FINITE ELEMENT METHOD

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**Abstract**

*Discretization of a continuous body containing an infinite number of points in the surface into a discrete model with a limited number of points (or nodes) is one of the fundamental steps in a finite element analysis. The shape of the body between these nodes is approximated by functions. These functions are known as shape functions and allow relationship of the coordinates of every point of a finite element with the positions of its nodes. The paper used theory of finite element to derive shape functions of some irregular elements that always appear in transitioning from one type of element to another in adaptive finite element analysis and likewise reduces higher degree elements to lower ones. Finally, the Lagrange family of shape functions were reduced to the Serendipity family via the same method.*

*Keywords: Lagrangian, serendipity family, discretization, finite element, interpolation*

## Introduction

A finite element approximation of displacements is given by

$$T(x) \approx \tilde{T} = \sum_i N_i(x) T_i = \boldsymbol{N}(x)\boldsymbol{T^e} \qquad (1)$$

where $N_i$ are element ***shape*** or ***interpolation*** functions, $T_i$ are nodal displacements and the sum ranges over the number of nodes associated with an element[3]
Shape functions in finite element analysis depend on the dimensionality of the problem and the type of elements used for discretization of the problem domain. The shape functions must also satisfy continuity requirements depending on the underlying PDE and form (strong or weak) of finite element formulation. Normally shape functions are chosen as polynomial functions. In addition, the shape functions must possess the following properties ([2];[5])

Partition of unity property:

$$\sum_{k=1}^{n} N_k = 1 \qquad (2)$$

Kronecker-delta property:

$$N_i(x_k) = \delta_{ik} \qquad (3)$$

where $n$ is the number of nodes in an element, $x_k$ represents the coordinates of $k^{th}$ node.

There are many ways of deriving shape functions of a regular element ([2];[1]). Irregular elements, such as transitions elements sometimes exhibit some peculiarities which can make the derivation of their shape functions a little tedious and time consuming using the general methods of the regular ones. In this work, irregular shape functions are derived using equation 1 and the existing regular elements shape functions. Also the higher order elements shape functions are reduced via same method to lower ones. Furthermore, a family of elements are also reduced to another family through the same method ([1], [4]).

## Transition from Linear to quadratics

Consider a transition from linear element to quadratic element as shown in the following diagram,

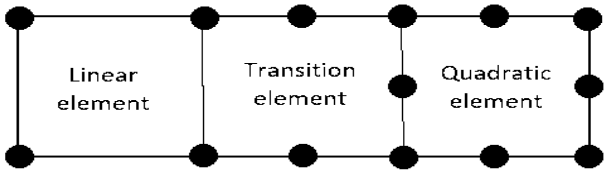


Figure 1: Transition from linear to quadratic showing 7-node transition element
There exist three sets of transition elements ranging from 5-node to 7-node as follows

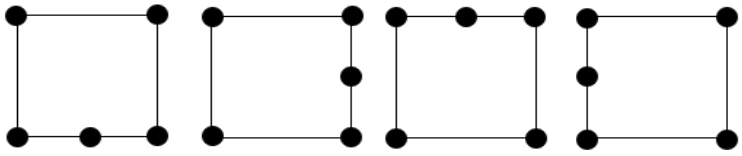



Figure 2: 5-node transition elements from linear to quadratic

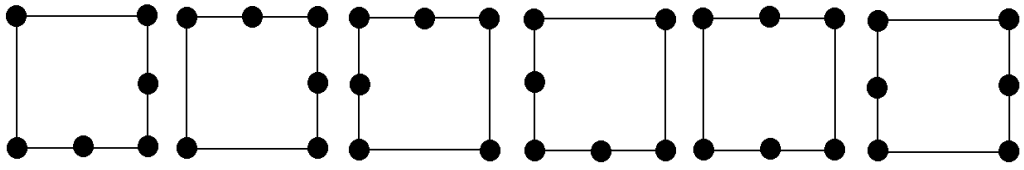



Figure 3: 6-node transition elements from linear to quadratic

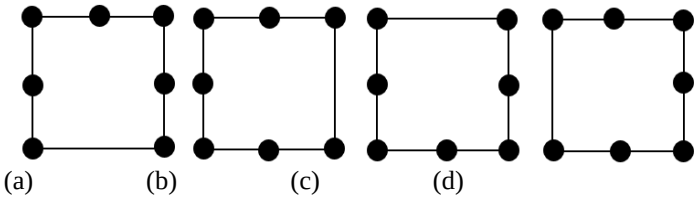



Figure 4: 7-node transition elements from linear to quadratic

The shape functions of these sets of transition elements can be derived via equation 1 together with regular 8-node quadratic element shape functions.

The idea is to use equation 1 to reduce the 8-node elements to any required irregular nodes element such as the sets depicted in the figures above.

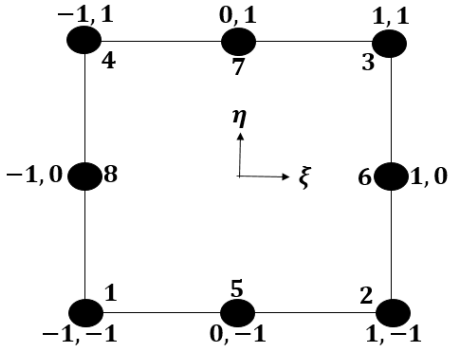



Figure 5: 8-node quadratic element

**Reduction from 8-node quadratic to 4-node linear element**

Consider the eight-node element shown in Figure 5. The element has four corner nodes and four mid-side nodes. The shape functions in the natural coordinates for the quadratic rectangular element are given as

$$\left.\begin{aligned} N_j &= \tfrac{1}{4}(1+\xi_j\xi)(1+\eta_j\eta)(\xi_j\xi+\eta_j\eta-1) \quad \text{for corner nodes } j=1,2,3,4 \\ N_j &= \tfrac{1}{2}(1+\xi^2)(1+\eta_j\eta) \quad \text{for corner nodes } j=5,7 \\ N_j &= \tfrac{1}{2}(1+\xi_j\xi)(1-\eta^2) \quad \text{for corner nodes } j=6,8 \end{aligned}\right\} \qquad (4)$$

where $(\xi_j, \eta_j)$ are the natural coordinates of node $j$.

From equation 4, the following eight shape functions are obtained

$$N_1(\xi,\eta)=-\frac{1}{4}(1-\xi)(1-\eta)(1+\xi+\eta) \quad N_2(\xi,\eta)=-\frac{1}{4}(1+\xi)(1-\eta)(1-\xi+\eta)$$
$$N_3(\xi,\eta)=-\frac{1}{4}(1+\xi)(1+\eta)(1-\xi-\eta) \quad N_4(\xi,\eta)=-\frac{1}{4}(1-\xi)(1+\eta)(1+\xi-\eta)$$
$$N_5(\xi,\eta)=\frac{1}{2}(1-\xi)(1+\xi)(1-\eta) \quad N_6(\xi,\eta)=\frac{1}{2}(1+\xi)(1+\eta)(1-\eta)$$
$$N_7(\xi,\eta)=\frac{1}{2}(1-\xi)(1+\xi)(1+\eta) \quad N_8(\xi,\eta)=\frac{1}{2}(1-\xi)(1+\eta)(1-\eta)$$

It is very easy to see that the shape functions possess the delta function property (equation 2) and partition of unity property (equation 3).

From equation 1, for 8-node quadratic element in figure 5,

$$T = N_j T_j \qquad j = 1,\ldots,8 \tag{5}$$

Thus,

$$T = N_1T_1 + N_2T_2 + N_3T_3 + N_4T_4 + N_5T_5 + N_6T_6 + N_7T_7 + N_8T_8 \tag{6}$$

Reducing first to 4-node linear element, the 4 side nodes of the quadratic element in figure 5 can be removed resulting to 4-node linear element in figure 6 by transforming equation 4 as follow,

$$T = N_1T_1 + N_2T_2 + N_3T_3 + N_4T_4 + N_5\left(\frac{T_1+T_2}{2}\right) + N_6\left(\frac{T_2+T_3}{2}\right) + N_7\left(\frac{T_3+T_4}{2}\right) + N_8\left(\frac{T_1+T_4}{2}\right) \tag{7}$$

$$T = \left(N_1 + \frac{N_5}{2} + \frac{N_8}{2}\right)T_1 + \left(N_2 + \frac{N_5}{2} + \frac{N_6}{2}\right)T_2 + \left(N_3 + \frac{N_6}{2} + \frac{N_7}{2}\right)T_3 + \left(N_4 + \frac{N_7}{2} + \frac{N_8}{2}\right)T_4 \tag{8}$$

Substituting the value of $N_j s$ in equation 8 and making the coefficient of $T_j s$ to be $n_i s, i, j = 1 \ldots 4$ gives,

$$n_1 = N_1 + \frac{N_5}{2} + \frac{N_8}{2}$$
$$n_1 = \left(-\frac{1}{4}(1-\xi)(1-\eta)(1+\xi+\eta) + \frac{1}{2}\left(\frac{(1-\xi)(1+\xi)(1-\eta)}{2}\right) + \frac{1}{2}\left(\frac{(1-\xi)(1+\eta)(1-\eta)}{2}\right)\right)$$
$$= \frac{1}{4}(\xi-1)(1-\eta).$$
$$n_2 = N_2 + \frac{N_5}{2} + \frac{N_6}{2}$$
$$n_2 = \left(-\frac{1}{4}(1+\xi)(1-\eta)(1-\xi+\eta) + \frac{1}{2}\left(\frac{(1-\xi)(1+\xi)(1-\eta)}{2}\right) + \frac{1}{2}\left(\frac{(1+\xi)(1+\eta)(1-\eta)}{2}\right)\right)$$
$$= \frac{1}{4}(\xi+1)(1-\eta)$$
$$n_3 = N_3 + \frac{N_6}{2} + \frac{N_7}{2}$$
$$n_3 = \left(-\frac{1}{4}(1+\xi)(1+\eta)(1-\xi-\eta) + \frac{1}{2}\left(\frac{(1+\xi)(1+\xi)(1-\eta)}{2}\right) + \frac{1}{2}\left(\frac{(1-\xi)(1+\xi)(1+\eta)}{2}\right)\right)$$
$$= \frac{1}{4}(1-\xi)(1+\eta)$$
$$n_4 = N_4 + \frac{N_7}{2} + \frac{N_8}{2}$$
$$n_4 = \left(-\frac{1}{4}(1-\xi)(1+\eta)(1+\xi-\eta) + \frac{1}{2}\left(\frac{(1-\xi)(1+\xi)(1+\eta)}{2}\right) + \frac{1}{2}\left(\frac{(1-\xi)(1+\eta)(1-\eta)}{2}\right)\right)$$
$$= \frac{1}{4}(1+\xi)(1+\eta)$$

where, $n_1$, $n_2$, $n_3$,$n_4$ are the linear element shape functions

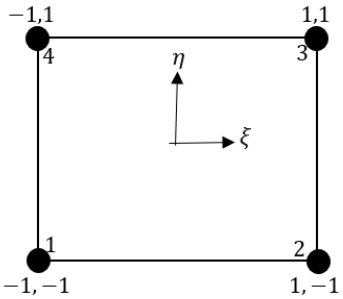



Figure 6: 4-node linear element

**Reduction from 8-node quadratic to 5-node transition element**

In the case of transition elements, the 5-node transition is used. Consider the 5-node transition element (a), having node on side one only, making five nodes with the mother nodes. Using equation 5, we have

$$T = N_1T_1 + N_2T_2 + N_3T_3 + N_4T_4 + N_5T_5 + N_6\left(\frac{T_2+T_3}{2}\right) + N_7\left(\frac{T_3+T_4}{2}\right) + N_8\left(\frac{T_1+T_4}{2}\right) \quad (9)$$

$$T = \left(N_1 + \frac{N_8}{2}\right)T_1 + \left(N_2 + \frac{N_6}{2}\right)T_2 + \left(N_3 + \frac{N_6}{2} + \frac{N_7}{2}\right)T_3 + \left(N_4 + \frac{N_7}{2} + \frac{N_8}{2}\right)T_4 + N_5T_5 \quad (10)$$

therefore,

$$n_1 = N_1 + \frac{N_8}{2}, \quad n_2 = N_2 + \frac{N_6}{2}, n_3 = N_3 + \frac{N_6}{2} + \frac{N_7}{2}, \quad n_4 = N_4 + \frac{N_7}{2} + \frac{N_8}{2}, n_5 = N_5$$

which gives the following shape functions upon substituting the values of $N_j s$ from the 8-nodes quadratic element.

$$n_1 = -\frac{1}{4}\xi(\xi-1)(\eta-1), \quad n_2 = -\frac{1}{4}\xi(\xi+1)(\eta-1), \quad n_3 = \frac{1}{4}(\xi+1)(\eta+1)$$

$$n_4 = -\frac{1}{4}(\xi-1)(\eta+1)n_5 = \frac{1}{2}(\xi-1)(\xi+1)(\eta-1)$$

The shape functions for the 6-node, 7-node and the remaining 5-node elements can be obtained using same procedure.

**Reduction from 12-node cubic elements to 4-node linear**

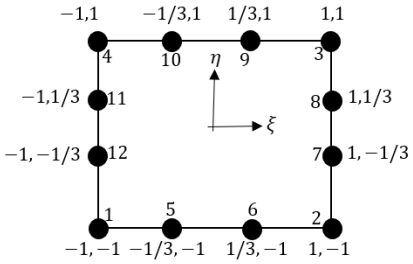



Figure 7: 12-node linear element

The shape functions for the 12-node serendipity cubic element shown in Figure 7 can be written as [3],

$$\left.\begin{aligned} N_j &= \frac{1}{32}(1+\xi_j\xi)(1+\eta_j\eta)(-10+(\xi^2+\eta^2)) \text{ for corner nodes } j = 1,2,3,4.\xi_j = \pm1 \text{ and } \eta_j = \pm1 \\ N_j &= \frac{9}{32}(1+\xi_j\xi)(1-\eta^2)(1+9\eta_j\eta) \text{ for side nodes } j = 7,8,11,12 \text{ where } \xi_j = \pm1 \text{ and } \eta_j = \pm\frac{1}{3} \\ N_j &= \frac{9}{32}(1+\eta_j\eta)(1-\xi^2)(1+9\xi_j\xi) \text{ for side nodes } j = 5,6,9,10 \text{ where } \xi_j = \pm\frac{1}{3} \text{ and } \eta_j = \pm1 \end{aligned}\right\} (11)$$

Substituting the values of $\xi_j$ and $\eta_j$ at each node of equation 11 gives the following twelve shape functions

$$N_1 = \frac{1}{32}(1-\xi)(1-\eta)\left(-10+9(\xi^2+\eta^2)\right), N_2 = \frac{1}{32}(1+\xi)(1-\eta)\left(-10+9(\xi^2+\eta^2)\right)$$
$$N_3 = \frac{1}{32}(1+\xi)(1+\eta)\left(-10+9(\xi^2+\eta^2)\right),\ N_4 = \frac{1}{32}(1-\xi)(1+\eta)\left(-10+9(\xi^2+\eta^2)\right)$$
$$N_5 = \frac{9}{32}(1-\eta)(1-3\xi)(1-\xi^2),\quad N_6 = \frac{9}{32}(1-\eta)(1+3\xi)(1-\xi^2)$$
$$N_7 = \frac{9}{32}(1+\xi)(1-3\eta)(1-\eta^2),\quad N_8 = \frac{9}{32}(1+\xi)(1+3\eta)(1-\eta^2)$$
$$N_9 = \frac{9}{32}(1+\eta)(1+3\xi)(1-\xi^2),\quad N_{10} = \frac{9}{32}(1+\eta)(1-3\xi)(1-\xi^2)$$
$$N_{11} = \frac{9}{32}(1-\xi)(1+3\eta)(1-\eta^2),\quad N_{12} = \frac{9}{32}(1-\xi)(1-3\eta)(1-\eta^2)$$

Using equation 1, the 4-node linear element can be obtained following the same procedure used to reduce 8-node quadratic to 4-node linear element as follows.

$$T = N_1T_1 + N_2T_2 + N_3T_3 + N_4T_4 + N_5T_5 + \cdots + N_{12}T_{12} \quad (12)$$
$$T = \left(N_1 + \tfrac{2}{3}N_5 + \tfrac{1}{3}N_6 + \tfrac{1}{3}N_{11} + \tfrac{2}{3}N_{12}\right)T_1 + \left(N_2 + \tfrac{1}{3}N_5 + \tfrac{2}{3}N_6 + \tfrac{2}{3}N_7 + \tfrac{1}{3}N_8\right)T_2 \quad (12)$$
$$+\left(N_3 + \tfrac{1}{3}N_7 + \tfrac{2}{3}N_8 + \tfrac{2}{3}N_9 + \tfrac{1}{3}N_{10}\right)T_3 + \left(N_4 + \tfrac{1}{3}N_9 + \tfrac{2}{3}N_{10} + \tfrac{2}{3}N_{11} + \tfrac{1}{3}N_{12}\right)T_4 \quad (13)$$

thus,

$$n_1 = N_1 + \frac{2}{3}N_5 + \frac{1}{3}N_6 + \frac{1}{3}N_{11} + \frac{2}{3}N_{12}$$
$$n_2 = N_2 + \frac{1}{3}N_5 + \frac{2}{3}N_6 + \frac{2}{3}N_7 + \frac{1}{3}N_8$$
$$n_3 = N_3 + \frac{1}{3}N_7 + \frac{2}{3}N_8 + \frac{2}{3}N_9 + \frac{1}{3}N_{10}$$
$$n_4 = N_4 + \frac{1}{3}N_9 + \frac{2}{3}N_{10} + \frac{2}{3}N_{11} + \frac{1}{3}N_{12}$$

which gives the four shape functions of 4-node linear element. Following same procedure all transition elements shape functions between linear and cubic can be obtained.

It should be noted that a 16-node quartic element can also be reduced to a 4-node linear element and to all transition elements for transiting from linear to quartic element via this method.

**Rectangular elements with various nodes.**

Obtaining the shape functions of rectangular elements with various nodes by reducing from regular rectangular element is easier using the finite element theory as shown by the following examples.

### 3.1 12-node serendipity element to 8-node element with various nodes

Example 1

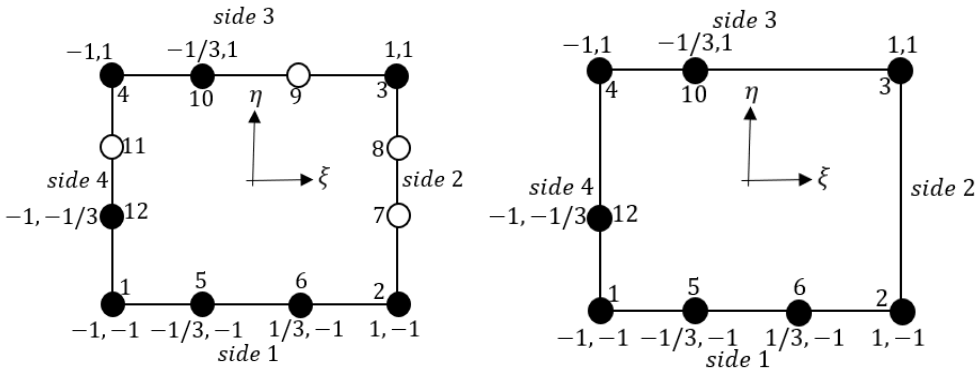



Figure 8: 12-node serendipity element to 8-node element with various nodes

Consider the elements in figure 8, to know the dimension of the element to reduce, we consider the coordinates of the nodes and pick the highest denominator, that is, if the highest denominator is $n$ then we have $n^{th}$ order. In this case, cubic element seems perfect, so we reduced a cubic element (a) to the required element (b) in figure 8 by the same procedure used in the previous reductions, which after all manipulations gives.

$$T = \left(N_1 + \frac{1}{3}N_{11}\right)T_1 + \left(N_2 + \frac{2}{3}N_7 + \frac{1}{3}N_8\right)T_2 + \left(N_3 + \frac{1}{3}N_7 + \frac{2}{3}N_8 + \frac{2}{3}N_9\right)T_3 + \left(N_4 + \frac{1}{3}N_9 + \frac{2}{3}N_{11}\right)T_4 + N_5T_5 + N_6T_6 + N_{10}T_{10} + N_{12}T_{12} \quad (14)$$

thus,

$$n_1 = N_1 + \frac{1}{3}N_{11}, \quad n_2 = N_2 + \frac{2}{3}N_7 + \frac{1}{3}N_8, \quad n_3 = N_3 + \frac{1}{3}N_7 + \frac{2}{3}N_8 + \frac{2}{3}N_9$$

$$n_4 = N_4 + \frac{1}{3}N_9 + \frac{2}{3}N_{11}, \quad n_5 = N_5, \quad n_6 = N_6, \quad n_7 = N_{10}, \quad n_8 = N_{12}$$

yielding,

$$N_1 = \frac{1}{32}(\xi - 1)(\eta - 1)(18\eta^2 + 12\eta + 9\xi^2 - 7),$$

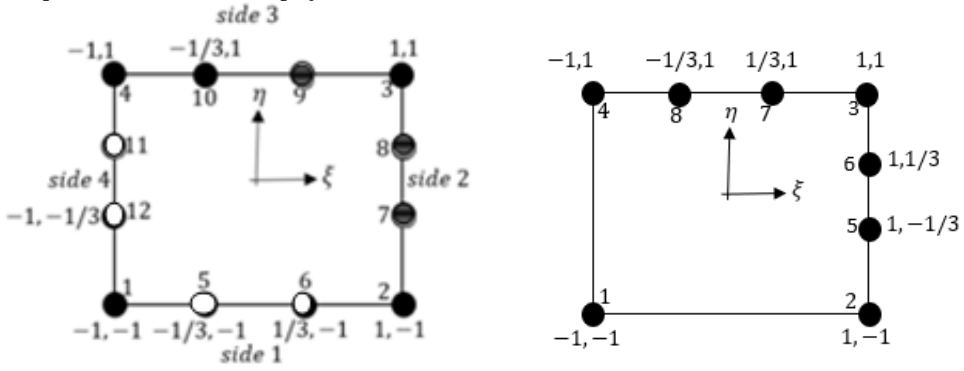
$$N_2 = -\frac{1}{32}(3\xi - 1)(1 + 3\xi)(1 + \xi)(\eta - 1), \; N_3 = -\frac{1}{32}(3\xi - 5)(1 + 3\xi)(1 + \xi)(\eta + 1)$$

$$N_4 = \frac{1}{32}(\xi - 1)(\eta + 1)(9\eta^2 - 12\eta + 18\xi^2 - 12\xi + 1)$$

$$N_5 = \frac{9}{32}(1 - \eta)(1 - 3\xi)(1 - \xi^2), \quad N_6 = \frac{9}{32}(1 - \eta)(1 + 3\xi)(1 - \xi^2)$$

$$N_7 = \frac{9}{32}(\eta + 1)(1 - 3\xi)(1 - \xi^2), \quad N_8 = \frac{9}{32}(1 - \xi)(1 - 3\eta)(1 - \eta^2)$$

Example 2: 12-node serendipity element to 8-node transition element




(a) (b)

Figure 9: **(a)** 12-node serendipity element (b) 8-node transition element

$$T = N_1T_1 + N_2T_2 + N_3T_3 + N_4T_4 + N_5T_5 + \cdots + N_{12}T_{12} \quad (15)$$

$$T = \left(N_1 + \frac{2}{3}N_5 + \frac{1}{3}N_6 + \frac{1}{3}N_{11} + \frac{2}{3}N_{12}\right)T_1 + \left(N_2 + \frac{1}{3}N_5 + \frac{2}{3}N_6\right)T_2 + (N_3)T_3 + \left(N_4 + \frac{2}{3}N_{11} + \frac{1}{3}N_{12}\right)T_4 \quad (16)$$

thus,

$$n_1 = N_1 + \frac{2}{3}N_5 + \frac{1}{3}N_6 + \frac{1}{3}N_{11} + \frac{2}{3}N_{12}, \quad n_2 = N_2 + \frac{1}{3}N_5 + \frac{2}{3}N_6, \; n_3 = N_3,$$

$$n_4 = N_4 + \frac{2}{3}N_{11} + \frac{1}{3}N_{12}, \quad n_5 = N_5, \quad n_6 = N_6, \quad n_7 = N_7, \quad n_8 = N_8$$

Example 3: 12-node serendipity element to 6-node element with various nodes

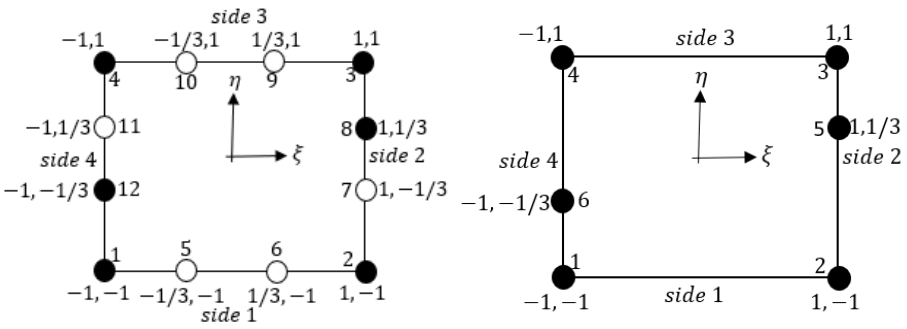


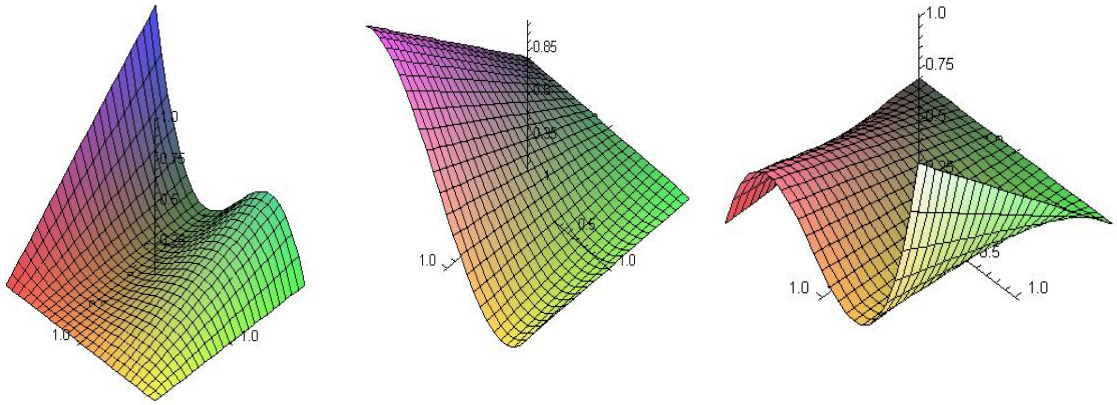

Figure 10: (a)12-node serendipity element, (b) 6-node element with various nodes

$$T = N_1T_1 + N_2T_2 + N_3T_3 + N_4T_4 + N_5T_5 + \cdots + N_{12}T_{12} \quad (17)$$

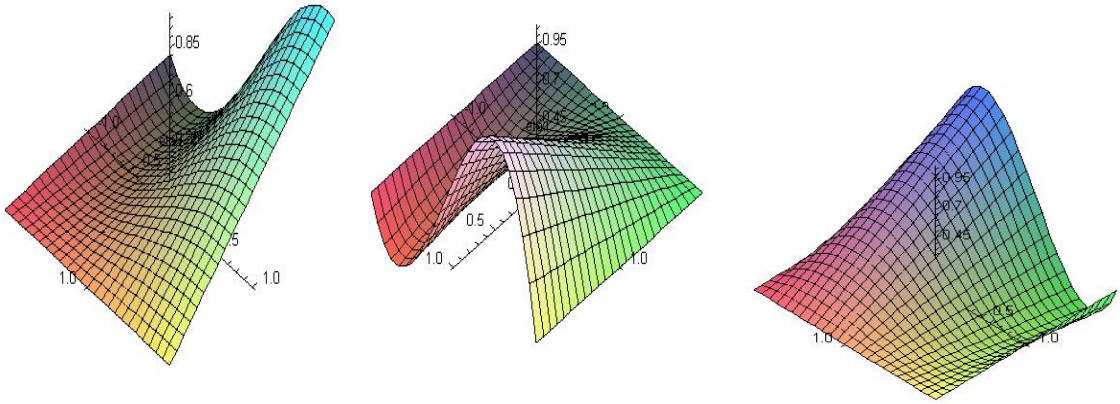
$$T = \left(N_1 + \frac{2}{3}N_5 + \frac{1}{3}N_6 + \frac{1}{3}N_{11}\right)T_1 + \left(N_2 + \frac{1}{3}N_5 + \frac{2}{3}N_6 + \frac{2}{3}N_7\right)T_2$$
$$+\left(N_3 + \frac{2}{3}N_9 + \frac{1}{3}N_{10} + \frac{1}{3}N_7\right)T_3 + \left(N_4 + \frac{2}{3}N_{11} + \frac{2}{3}N_{10} + \frac{1}{3}N_9\right)T_4$$
$$+(N_8)T_8 + (N_{12})T_{12} \quad (18)$$

thus,

$$n_1 = N_1 + \frac{2}{3}N_5 + \frac{1}{3}N_6 + \frac{1}{3}N_{11} = \frac{1}{16}(\eta - 1)(1 + 3\eta)^2(-1 + \xi)$$
$$n_2 = N_2 + \frac{1}{3}N_5 + \frac{2}{3}N_6 + \frac{2}{3}N_7 = \frac{1}{32}(3\eta + 5)(\eta - 1)(3\eta - 1)(1 + \xi)$$
$$n_3 = N_3 + \frac{2}{3}N_9 + \frac{1}{3}N_{10} + \frac{1}{3}N_7 = \frac{1}{16}(\eta + 1)(3\eta - 1)^2(1 + \xi)$$
$$n_4 = N_4 + \frac{2}{3}N_{11} + \frac{2}{3}N_{10} + \frac{1}{3}N_9 = \frac{1}{32}(3\eta - 5)(1 + 3\eta)(1 + \eta)(-1 + \xi)$$
$$n_5 = N_8 = \frac{9}{32}(1 + \xi)(1 + 3\eta)(1 - \eta^2)$$
$$n_6 = N_{12} = \frac{9}{32}(1 - \xi)(1 - 3\eta)(1 - \eta^2)$$




$n_1 n_2 n_3$



$n_4 n_5 n_6$

Figure 11: Shape functions ($n_1, n_2, n_3, n_4, n_5,$ and $n_6$) of the element in figure 10 (b)

## 4 Lagrange element to serendipity element

Some Lagrange elements can be converted to serendipity element through simple modification of their shape functions; in this section two Lagrange elements are considered: quadratic and cubic elements.

### 4.1 Quadratic Lagrange 9-node element to quadratic serendipity 8-node element

The shape function for quadratic element of the Lagrange family can be derived using the following formulas [3]

$$\left.\begin{aligned} N_j &= \tfrac{1}{4}(1+\xi_j\xi)\xi_j\xi(1+\eta_j\eta)\eta_j\eta \quad j=1,2,3,4, \quad \xi_j=\pm1, \eta_j=\pm1 \\ N_j &= \tfrac{1}{2}(1-\xi^2)(1+\eta_j\eta)\eta_j\eta \quad j=5,7, \quad \xi_j=0, \eta_j=\pm1 \\ N_j &= \tfrac{1}{2}(1-\eta^2)(1+\xi_j\xi)\xi_j\xi \quad j=6,8, \quad \xi_j=\pm1, \ \eta_j=0 \\ N_j &= (1-\xi^2)(1+\eta^2) \quad j=9 \quad \xi_j=0, \ \eta_j=0 \end{aligned}\right\} \qquad (19)$$

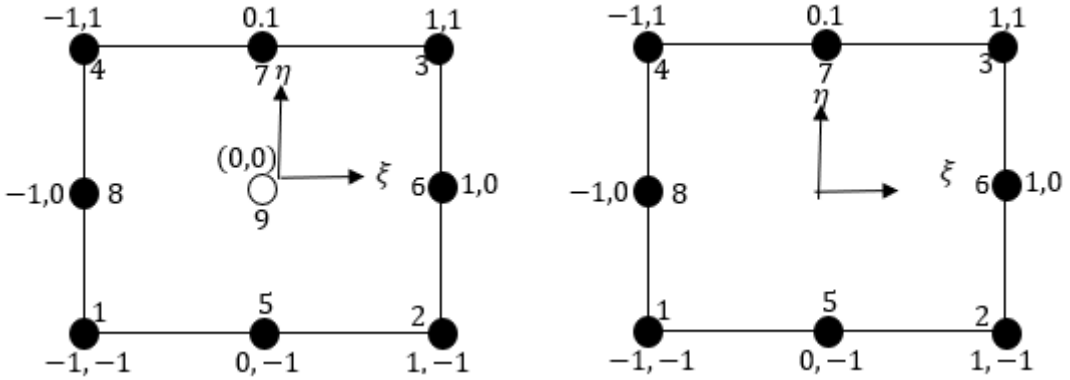



**Figure 12:** 9-node Lagrange element to 8-node serendipity element.

From equation 1, considering figure 13(a) we have,

$$T = N_1T_1 + N_2T_2 + N_3T_3 + N_4T_4 + N_5T_5 + \cdots + N_9T_9 \qquad (20)$$

To remove the inner node of the element (node 9), ½ of shape function of the 9th node must be added to all the side nodes (5,6,7,8) and at the same time ¼ of it must be subtracted from the corner nodes(1,2,3,4) which gives,

$$T = \left(N_1 - \frac{1}{4}N_9\right)T_1 + \left(N_2 - \frac{1}{4}N_9\right)T_2 + \left(N_3 - \frac{1}{4}N_9\right)T_3 + \left(N_4 - \frac{1}{4}N_9\right)T_4 + \left(N_5 + \frac{1}{2}N_9\right)T_5$$
$$+ \left(N_6 + \tfrac{1}{2}N_9\right)T_6 + \left(N_7 + \tfrac{1}{2}N_9\right)T_7 + \left(N_8 + \tfrac{1}{2}N_9\right)T_8. \qquad (21)$$

Thus,

$$n_1 = \left(N_1 - \frac{1}{4}N_9\right), \quad n_2 = \left(N_2 - \frac{1}{4}N_9\right),$$
$$n_3 = \left(N_3 - \frac{1}{4}N_9\right), \quad n_4 = \left(N_4 - \frac{1}{4}N_9\right),$$
$$n_5 = \left(N_5 + \frac{1}{2}N_9\right), \quad n_6 = \left(N_6 + \frac{1}{2}N_9\right),$$
$$n_7 = \left(N_7 + \frac{1}{2}N_9\right), \quad n_8 = \left(N_8 + \frac{1}{2}N_9\right)$$

**Cubic Lagrange 16-node element to cubic serendipity 12-node element**

The cubic Lagrange element in figure 12(a) is to be reduced to the serendipity element of figure 12(b) via equation 1.

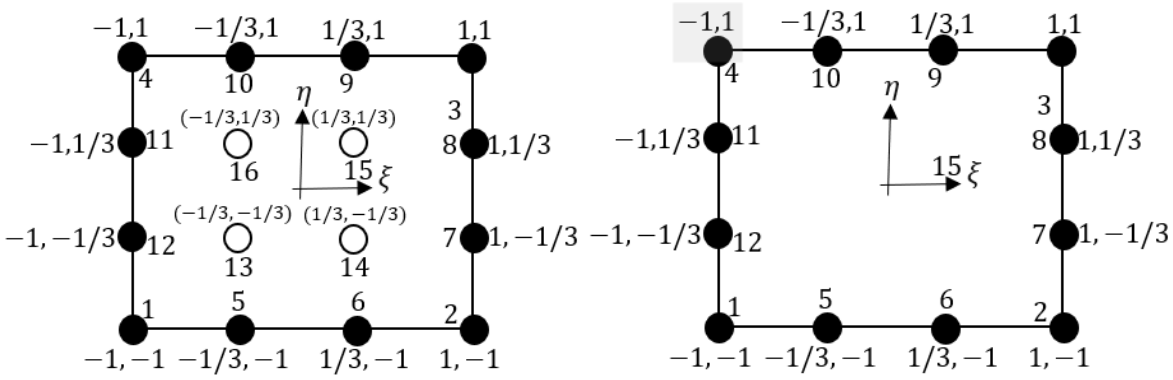



Figure 13:16-node Lagrange element to 12-node serendipity element

Using equation 1, we have,

$$T = N_1T_1 + N_2T_2 + N_3T_3 + N_4T_4 + N_5T_5 + \cdots + N_{16}T_{16} \qquad (22)$$

$$T = \left(N_1 - \frac{1}{4}(L1)(N13 + N14 + N15 + N16)\right)T_1$$
$$+\left(N_2 - \frac{1}{4}(L2)(N13 + N14 + N15 + N16)\right)T_2$$
$$+\left(N_3 - \frac{1}{4}(L3)(N13 + N14 + N15 + N16)\right)T_3$$
$$+\left(N_4 - \frac{1}{4}(L4)(N13 + N14 + N15 + N16)\right)T_4$$
$$+\left(N_5 + \frac{2}{3}N13 + \frac{1}{3}N16\right)T_5 + \left(N_6 + \frac{2}{3}N14 + \frac{1}{3}N15\right)T_6 + \left(N_7 + \frac{1}{3}N13 + \frac{2}{3}N14\right)T_7$$
$$+\left(N_8 + \frac{2}{3}N15 + \frac{1}{3}N16\right)T_8 + \left(N_9 + \frac{1}{3}N14 + \frac{2}{3}N15\right)T_9 + \left(N_{10} + \frac{1}{3}N13 + \frac{1}{3}N16\right)T_{10}$$
$$+\left(N_{11} + \frac{2}{3}N16 + \frac{1}{3}N15\right)T_{11} + \left(N_{12} + \frac{2}{3}N13 + \frac{1}{3}N14\right)T_{12} \qquad (23)$$

where, $L_1, L_2, L_3$ *and* $L_4$ are the linear shape functions, thus,

$$n_1 = N_1 - \frac{1}{4}(L_1)(N13 + N14 + N15 + N16) = \frac{1}{32}(1-\xi)(1-\eta)\left(-10 + 9(\xi^2 + \eta^2)\right)$$
$$n_2 = N_2 - \frac{1}{4}(L_2)(N13 + N14 + N15 + N16) = \frac{1}{32}(1+\xi)(1-\eta)\left(-10 + 9(\xi^2 + \eta^2)\right)$$
$$n_3 = N_3 - \frac{1}{4}(L_3)(N13 + N14 + N15 + N16) = \frac{1}{32}(1+\xi)(1+\eta)\left(-10 + 9(\xi^2 + \eta^2)\right)$$
$$n_4 = N_4 - \frac{1}{4}(L_4)(N13 + N14 + N15 + N16) = \frac{1}{32}(1-\xi)(1+\eta)\left(-10 + 9(\xi^2 + \eta^2)\right)$$
$$n_5 = N_5 + \frac{2}{3}N13 + \frac{1}{3}N16 = \frac{9}{32}(1-\eta)(1-3\xi)(1-\xi^2)$$
$$n_6 = N_6 + \frac{2}{3}N14 + \frac{1}{3}N15 = \frac{9}{32}(1-\eta)(1+3\xi)(1-\xi^2)$$

$$n_7 = N_7 + \frac{1}{3}N13 + \frac{2}{3}N14 = \frac{9}{32}(1+\xi)(1-3\eta)(1-\eta^2)$$
$$n_8 = N_8 + \frac{2}{3}N15 + \frac{1}{3}N16 = \frac{9}{32}(1+\xi)(1+3\eta)(1-\eta^2)$$
$$n_9 = N_9 + \frac{1}{3}N14 + \frac{2}{3}N15 = \frac{9}{32}(1+\eta)(1+3\xi)(1-\xi^2)$$
$$n_{10} = N_{10} + \frac{1}{3}N13 + \frac{1}{3}N16 = \frac{9}{32}(1+\eta)(1-3\xi)(1-\xi^2)$$
$$n_{11} = N_{11} + \frac{2}{3}N16 + \frac{1}{3}N15 = \frac{9}{32}(1-\xi)(1+3\eta)(1-\eta^2)$$
$$n_{12} = N_{12} + \frac{2}{3}N13 + \frac{1}{3}N14 = \frac{9}{32}(1-\xi)(1-3\eta)(1-\eta^2)$$

**Conclusion**

The paper discussed the use of finite element theory to manipulate shape functions of different orders. Shape function been a very crucial part of finite element method may be difficult to derive especially when it concerns the transition from one order to another such as in linear to quadratic, linear to quartic etc, encounter when dealing with adaptive finite element analysis. It is hoped that the method will be a good tool for transitioning smoothly from one type of element to another.

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